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THE CONVECTED DESCRIPTION IN LARGE DEFORMATION FRICTIONAL CONTACT PROBLEMS

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Abstract—The convected description is introduced for finite deformation frictional contact problems, such that both Coulomb and more general friction criteria can be conveniently and reliably numerically implemented in Lagrangian finite element calculations. The discussion includes careful definition of the convected frame from the standpoint of differential geometry, complete with physical interpretations of the frame and the associated slip and traction rate measures. As demonstrated, use of these measures enables prescription of frictional constitutive relationships with virtually no more complexity than is required by the kinematically linear theory, while ensuring the required frame indifference of the description. As a result, the work provides a sound mathematical and continuum mechanical foundation for future implementations of general friction laws in large deformation solid mechanics computing architectures.

1. INTRODUCTION

Recent years have seen an increase in attempts to model large deformation contact and friction problems in solid mechanics using the Lagrangian description. These efforts have seemingly fallen into two primary categories. First, a series of works have been presented in which the formulation of friction proposed has been specific to a numerical method (usually the finite element method), and often to the specific numerical discretization scheme used (Ju and Taylor, 1988; Benson and Hallquist, 1990; Wriggers *et al.*, 1990; Rebelo *et al.*, 1990). Second, and somewhat more abstractly, works have been presented in which the problem has been attacked from the more general continuum mechanical view, with considerable attention paid to the manipulations necessary to maintain frame indifference in the large deformation setting (Kikuchi and Oden, 1988 or Glaser, 1992). In the point of view of the author, a shortcoming of the former body of work is a lack of extendibility to varying discretization schemes and modeling situations. The latter body of work has relied heavily on somewhat contrived objective rates from the theory of finite plasticity, resulting in frictional equations that are heavily couched in covariant derivatives and other objects from tensor calculus, making the theory unwieldly to implement numerically.

The objective of the author's research has been to develop a formulation of large deformation frictional contact, with a rational continuum mechanical basis, that is comparatively simple to implement in nonlinear finite element codes. In earlier work (Laursen, 1992; Laursen and Simo, 1993), such a framework was developed and demonstrated for the case of the Coulomb frictional problem. Through a careful treatment of contact kinematics, the virtual work of contact, and subsequent finite element discretization, a formulation was developed and demonstrated for both quasistatic and dynamic solid mechanics problems. This work, and its frame indifference, relies crucially on the use of a convected reference frame in which the frictional equations may be written, although the details of and justification for the use of this frame were largely omitted from these references. In this paper, this gap in the theory is bridged, by providing an analysis of this reference frame from the perspective of differential geometry. The necessary kinematic quantities for frictional characterization are developed within this frame, and their incorporation into a constitutive theory is demonstrated by using Coulomb friction as a model. It is felt that the physical insight gained in this analysis, supported by the underlying mathematics, provides the needed background for effective general implementations of friction. In particular, it is demonstrated that the convected frame used leads to extremely simple expressions for the tractions, in which covariant derivatives and complicated objective rates do not appear. The theory provided here would therefore appear to provide an unprecedented generality and simplicity for this otherwise very complex class of problems.

2. FRICTIONAL KINEMATIC FRAMEWORK

This section starts with a brief review of the notation used for characterization of the frictional contact problem, introduced previously in Laursen (1992) and Laursen and Simo (1993). This discussion is used as background for the main topic of this paper: an in-depth examination of the kinematics of a convected reference frame in which the equations of frictional contact are conveniently expressed. This frame has at least two strong advantages: (1) the objects needed for most theories of friction, relative velocity and objective stress rates, are easily described and understood from the point of view of differential geometry, and (2) these objects possess a convenient physical interpretation in this frame, making adaptation of experimentally-observed frictional behavior to the constitutive framework and subsequent numerical implementation much easier. The former point is investigated in this section, while the latter is discussed and demonstrated in Section 3.

2.1. *Definitions and notation*

In what follows the motions of two bodies, denoted $\Omega^{(1)}$ and $\Omega^{(2)}$, are considered [see Fig. 1]. These motions, which are to be determined over an interval of time $\mathbf{I} = [0, T]$, may be taken to be either quasistatic or dynamic in nature. These motions are assumed to induce mechanical contact between the two bodies during $\mathbf{I}$. Subsets of the bodies' surfaces $\partial\Omega^{(i)}$, $i = 1, 2$ are therefore designated as $\Gamma^{(i)}$, $i = 1, 2$, such that all prospective points of contact during $\mathbf{I}$ are included. The motions are considered to be mappings from the reference configurations to the ambient space $\mathbb{R}^{n_{sd}}$ via:

$$\boldsymbol{\varphi}^{(i)} : \bar{\Omega}^{(i)} \times \mathbf{I} \rightarrow \mathbb{R}^{n_{sd}}; \quad i = 1, 2, \tag{1}$$

where n_{sd} is the number of spatial dimensions. In formulating the contact conditions, the primary problem is to relate the contact tractions generated at points $\mathbf{X} \in \Gamma^{(1)}$ and $\mathbf{Y} \in \Gamma^{(2)}$ to these motions.

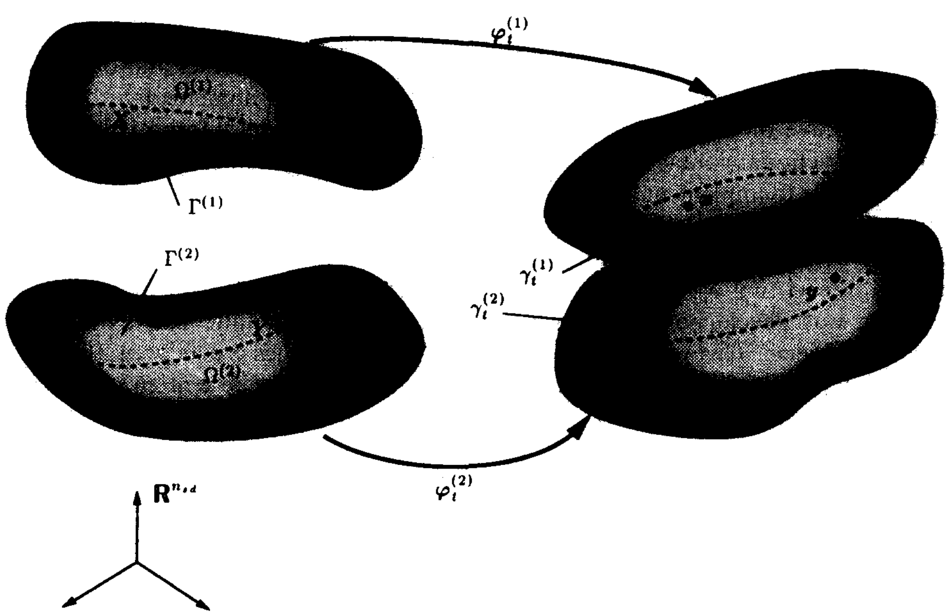

Fig. 1. Basic notation for the large deformation frictional contact problem.

2.1.1. *Closest point projection. Normal contact conditions.* The configuration of a body (i) at time t, obtained by fixing the time variable in the motion $\boldsymbol{\varphi}^{(i)}$, is written as $\boldsymbol{\varphi}_t^{(i)}$. The collection of all such configurations during $\mathbf{I}$ is therefore a curve in $\mathbb{R}^{n_{sd}}$. At a given time $t \in \mathbf{I}$, the current positions of the contact surfaces are written as $\gamma_t^{(i)} = \boldsymbol{\varphi}_t^{(i)}(\Gamma^{(i)})$, $i = 1, 2$. Picking a point $\mathbf{X} \in \Gamma^{(1)}$, the following gap or gauge function can be defined in terms of the closest point projection of $\boldsymbol{\varphi}_t^{(1)}(\mathbf{X})$ onto $\gamma_t^{(2)}$:

$$\left.\begin{aligned} g(\mathbf{X}, t) &= \operatorname{sign}(g(\mathbf{X}, t))|g(\mathbf{X}, t)|, \quad \text{where} \\ |g(\mathbf{X}, t)| &= \min_{\mathbf{Y} \in \Gamma^{(2)}} \|\boldsymbol{\varphi}^{(1)}(\mathbf{X}, t) - \boldsymbol{\varphi}^{(2)}(\mathbf{Y}, t)\|, \quad \text{and} \\ \operatorname{sign}(g(\mathbf{X}, t)) &= \begin{cases} -1 \text{ if } \boldsymbol{\varphi}^{(1)}(\mathbf{X}, t) \text{ is admissible} \\ \quad 1 \text{ otherwise.} \end{cases} \end{aligned}\right\} \tag{2}$$

Clearly, interpenetration of the two bodies is precluded locally by requiring that $g(\mathbf{X}, t) \leqslant 0$. The simplest expression of the normal pressure-gap relationship is that given by the Kuhn–Tucker conditions, specified by:

$$\left.\begin{aligned} t_N(\mathbf{X}, t) &\geqslant 0 \\ g(\mathbf{X}, t) &\leqslant 0 \\ t_N(\mathbf{X}, t) g(\mathbf{X}, t) &= 0 \end{aligned}\right\}, \tag{3}$$

where t_N is the contact pressure at $\mathbf{X}$. These conditions can be converted to a deterministic form via introduction of a normal penalty ε_N, rendering the penalized form:

$$t_N(\mathbf{X}, t) = \varepsilon_N g^+(\mathbf{X}, t), \tag{4}$$

where g^+ is the positive part of g. Other relationships between the gap (or approach) g and the pressure t_N are also possible, depending on the theory used to characterize the micromechanics of the contact at $\mathbf{X}$ (see, for example, Wriggers and Zavarise, 1993). In any case, it is assumed without much loss of generality that the expression for t_N is of the form:

$$t_N(\mathbf{X}, t) = f(g^+(\mathbf{X}, t)), \tag{5}$$

where $f(\cdot)$ is a (possibly nonlinear) continuous algebraic function of g^+, with the property that $f(0) = 0$.

2.1.2. *Underlying parametrization. Frictional kinematics.* An important aspect of the current formulation is that the contact traction associated with a given contact point $\mathbf{X} \in \Gamma^{(1)}$ (or conversely, $\mathbf{Y} \in \Gamma^{(2)}$) is resolved and described in terms of the opposing surface $\gamma_t^{(2)}$ $(\gamma_t^{(1)})$. Accordingly, an underlying parametrization is assumed for this surface. In particular,

$$\begin{aligned} \Gamma^{(2)} &= \boldsymbol{\Psi}_0^{(2)}(\mathscr{A}^{(2)}) \quad \text{and} \\ \gamma_t^{(2)} &= \boldsymbol{\Psi}_t^{(2)}(\mathscr{A}^{(2)}), \end{aligned} \tag{6}$$

where $\mathscr{A}^{(2)} \subset \mathbb{R}^{n_{sd}-1}$ and the mappings $\boldsymbol{\Psi}_0^{(2)}$ and $\boldsymbol{\Psi}_t^{(2)}$ are assumed smooth for simplicity [see Fig. 2]. One may notice that as a result of definitions (6), $\boldsymbol{\Psi}_t^{(2)} = \boldsymbol{\varphi}_t^{(2)} \circ \boldsymbol{\Psi}_0^{(2)}$. Typical points of $\mathscr{A}^{(2)}$ are denoted as $\boldsymbol{\xi}$.

The point of $\Gamma^{(2)}$ achieving the minimization indicated in $(2)_2$ for a given $\mathbf{X} \in \Gamma^{(1)}$ and at an instant $t \in \mathbf{I}$ is denoted as $\bar{\mathbf{Y}}(\mathbf{X}, t)$. This notation asserts that although points $\mathbf{Y} \in \Gamma^{(2)}$ are independent variables of the problem on the same footing as t and $\mathbf{X} \in \Gamma^{(1)}$, identification of the particular point $\bar{\mathbf{Y}}$ depends on both through the motions and the closest point projection in $(2)_2$. The point of $\mathscr{A}^{(2)}$ corresponding to $\bar{\mathbf{Y}}(\mathbf{X}, t)$ is similarly denoted $\bar{\boldsymbol{\xi}}(\mathbf{X}, t)$.

Both the surface $\Gamma^{(2)}$ and its current configuration $\gamma_t^{(2)}$ can be considered to be $(n_{sd} - 1)$-

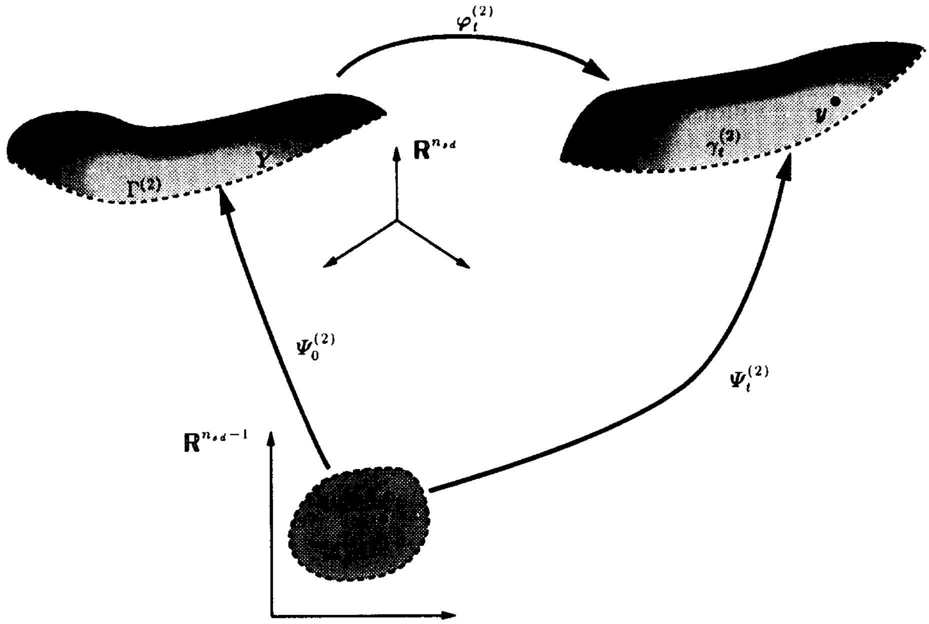

Fig. 2. Schematic of the parametrization of a contact surface.

dimensional manifolds in $\mathbb{R}^{n_{sd}}$, with coordinate systems over each given by the parametrizations $\Psi_0^{(2)}$ and $\Psi_t^{(2)}$, respectively. As such, coordinate base vectors $\mathbf{E}_\alpha$ and $\mathbf{e}_\alpha$ for these manifolds can be defined via:

$$\begin{aligned} \mathbf{E}_\alpha &:= \Psi^{(2)}_{0,\alpha}, \quad \text{and} \\ \mathbf{e}_\alpha &:= \Psi^{(2)}_{t,\alpha}, \quad \alpha = 1,\ldots,n_{sd}-1. \end{aligned} \tag{7}$$

In (7) and in all subsequent development, lowercase Greek indices α, β, γ, etc. are reserved for quantities expressed in these bases.

Since the objective of the current formulation is to write the frictional governing equations for a point $\mathbf{X} \in \Gamma^{(1)}$ opposing this surface, it will prove convenient to define a basis, associated with each point $\mathbf{X}$, which is convected with that point as it moves. The following definitions are therefore made:

$$\begin{aligned} \mathbf{T}_\alpha &:= \mathbf{E}_\alpha(\bar{\xi}(\mathbf{X},t)), \quad \text{and} \\ \boldsymbol{\tau}_\alpha &:= \mathbf{e}_\alpha(\bar{\xi}(\mathbf{X},t)), \quad \alpha = 1,\ldots,n_{sd}-1. \end{aligned} \tag{8}$$

In (8) and the rest of the paper, the arguments of $\mathbf{T}_\alpha$ and $\boldsymbol{\tau}_\alpha$ are suppressed, with the understanding that they are always associated with a material point $\mathbf{X}$ and are evaluated according to its current projection, parametrized by $\bar{\xi}(\mathbf{X},t)$.

Finally, in characterizing the frictional response it is useful to write the relative velocity between material point $\mathbf{X}$ and the adjacent surface in terms of the convected basis introduced in (8). To do so one considers the case where $\mathbf{X}$ is sliding on the opposing surface, such that $g(\mathbf{X},t) = 0$ and no separation is occurring. Upon reflection one realizes that these conditions imply

$$\dot{g}(\mathbf{X},t) = 0, \tag{9}$$

which, in view of $(2)_2$, leads to:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{\varphi}^{(1)}(\mathbf{X},t) - \boldsymbol{\varphi}^{(2)}(\bar{\mathbf{Y}}(\mathbf{X},t),t)] \\ &= \mathbf{V}^{(1)}(\mathbf{X},t) - \mathbf{V}^{(2)}(\bar{\mathbf{Y}}(\mathbf{X},t),t) - \mathbf{F}_t^{(2)}(\Psi_0^{(2)}(\bar{\xi}(\mathbf{X},t)))\frac{\mathrm{d}}{\mathrm{d}t}[\bar{\mathbf{Y}}(\mathbf{X},t)], \end{aligned} \tag{10}$$

where $\mathbf{F}_t^{(2)}$ is the deformation gradient in body (2) at time t. The quantity $\mathbf{V}^{(1)}(\mathbf{X},t)-\mathbf{V}^{(2)}(\bar{\mathbf{Y}}(\mathbf{X},t),t)$ is recognized as the difference of the material velocities associated with the points $\mathbf{X}\in\Gamma^{(1)}$ and $\bar{\mathbf{Y}}\in\Gamma^{(2)}$, and therefore represents the instantaneous tangential relative velocity. Equation (10) tells us that this physical quantity may be represented in terms of vectors in the tangent spaces of $\Gamma^{(2)}$ and $\gamma_t^{(2)}$. The vectors $\boldsymbol{\upsilon}_T$ and $\boldsymbol{v}_T$ are therefore defined as follows:

$$\boldsymbol{\upsilon}_T := \mathbf{F}_t^{(2)}(\boldsymbol{\Psi}_0^{(2)}(\bar{\xi}(\mathbf{X},t)))\frac{\mathrm{d}}{\mathrm{d}t}[\bar{\mathbf{Y}}(\mathbf{X},t)], \quad \text{and}$$

$$\boldsymbol{v}_T := \frac{\mathrm{d}}{\mathrm{d}t}[\bar{\mathbf{Y}}(\mathbf{X},t)] = \dot{\bar{\xi}}^{\alpha}(\mathbf{X},t)\mathbf{T}_{\alpha}. \tag{11}$$

Examination of (11) will reveal that $\boldsymbol{v}_T$ represents the pull back of $\boldsymbol{\upsilon}_T$ by $\boldsymbol{\varphi}_t^{(2)}$. In fact,

$$\boldsymbol{\upsilon}_T = \dot{\bar{\xi}}^{\alpha}(\mathbf{X},t)\boldsymbol{\tau}_{\alpha}. \tag{12}$$

2.2. *The convected description*

In formulating the frictional contact conditions one has a choice: whether to cast the equations in terms of $\boldsymbol{\upsilon}_T$ in $(11)_1$ or $\boldsymbol{v}_T$ in $(11)_2$. The quantity $\boldsymbol{\upsilon}_T$, although associated with the material point $\mathbf{X}\in\Gamma^{(1)}$, physically represents a vector expressed in spatial coordinates. Likewise, considering the Piola (nominal) contact traction:

$$\mathbf{t}(\mathbf{X},t) := \mathbf{P}(\mathbf{X},t)\cdot\mathbf{n}_0(\mathbf{X}), \tag{13}$$

where $\mathbf{P}$ is the first Piola–Kirchhoff stress and $\mathbf{n}_0$ is the outward surface normal in the reference configuration, one finds that $\mathbf{t}$ is another object that can be thought of as a vector in the spatial frame. Therefore, a constitutive law in which $\boldsymbol{\upsilon}_T$ and the frictional part of $\mathbf{t}$ appear as primary variables is in a sense a spatial representation of the frictional contact problem. It would appear that spatial descriptions of one form or another have been used in virtually all previous treatments of this problem (Kikuchi and Oden, 1988; Wriggers *et al.*, 1990 or Glaser, 1992).

Equation $(11)_2$, however, suggests that another alternative exists. Following a definition given in several treatments of nonlinear continuum mechanics (Marsden and Hughes, 1983), $\boldsymbol{v}_T$ is defined as the convective relative velocity associated with the point $\mathbf{X}\in\Gamma^{(1)}$. As indicated in $(11)_2$, $\boldsymbol{v}_T$ is a vector based at $\bar{\mathbf{Y}}(\mathbf{X},t)$ which lies in the tangent space of $\Gamma^{(2)}$. Examination of eqns (11) will reveal that the components of $\boldsymbol{v}_T$ in the $\mathbf{T}_\alpha$ basis are equal to those of $\boldsymbol{\upsilon}_T$ in the $\boldsymbol{\tau}_\alpha$ basis. Stated another way, the components of $\boldsymbol{v}_T$ are the same as those obtained by resolving $\boldsymbol{\upsilon}_T$ in a basis scribed on, or convected with, $\Gamma^{(2)}$ as it deforms. This property is a characteristic of such convected quantities (see again Marsden and Hughes, 1983), and gives rise in general to the terminology to be used herein, the convected description.

In order to work in this frame it is necessary to define a convected frictional stress. One may begin by resolving the Piola traction $\mathbf{t}(\mathbf{X},t)$ in the spatial frame to obtain the (spatial) frictional traction $\mathbf{t}_T$:

$$\mathbf{t}_T(\mathbf{X},t) := -\mathbf{t}(\mathbf{X},t) - t_N(\mathbf{X},t)\boldsymbol{v}, \tag{14}$$

where $\boldsymbol{v}$ is the outward normal to $\gamma_t^{(2)}$ at $\boldsymbol{\varphi}_t^{(2)}(\bar{\mathbf{Y}}(\mathbf{X},t))$ (not to be confused with the convected velocity $\boldsymbol{v}_T$) and t_N is the contact pressure, positive if compressive, mentioned in (3). Note that a sign change has been employed in the definition of $\mathbf{t}_T(\mathbf{X},t)$, so that it physically represents the frictional traction exerted by $\mathbf{X}$ on the surface $\Gamma^{(2)}$.

The vector $\mathbf{t}_T(\mathbf{X},t)$ lies in the tangent space of $\gamma_t^{(2)}$ at $\boldsymbol{\varphi}_t^{(2)}(\bar{\mathbf{Y}}(\mathbf{X},t))$, and can be resolved via:

$$\mathbf{t}_T(\mathbf{X}, t) = t_T^{\alpha}(\mathbf{X}, t)\boldsymbol{\tau}_{\alpha}. \tag{15}$$

As discussed previously, $\mathbf{t}_T$ is a spatial object. The convected frictional traction $\mathscr{T}_T(\mathbf{X}, t)$ is defined as the pull back of $\mathbf{t}_T$ induced by $\boldsymbol{\varphi}_t^{(2)}$:

$$\left.\begin{aligned}\mathscr{T}_T(\mathbf{X}, t) &:= \mathbf{F}_t^{(2)^{-1}}(\bar{\mathbf{Y}}(\mathbf{X}, t))\mathbf{t}_T(\mathbf{X}, t)\\ &= t_T^{\alpha}(\mathbf{X}, t)\mathbf{T}_{\alpha}\end{aligned}\right\}. \tag{16}$$

As was the case with $\mathbf{v}_T$, the components of $\mathscr{T}_T$ in the $\mathbf{T}_\alpha$ frame equal those of $\mathbf{t}_T$ in the $\boldsymbol{\tau}_\alpha$ frame. Of much use in the ensuing development will be the one-form $\mathscr{T}_T^b$ associated with the vector $\mathscr{T}_T$:

$$\mathscr{T}_T^b(\mathbf{X}, t) := t_{T_\alpha}(\mathbf{X}, t)\mathbf{T}^{\alpha}, \text{ where}$$
$$t_{T_\alpha}(\mathbf{X}, t) = M_{\alpha\beta}t_T^{\beta}(\mathbf{X}, t). \tag{17}$$

In $(17)_2$, $M_{\alpha\beta} = \mathbf{T}_\alpha \cdot \mathbf{T}_\beta$ is the metric associated with the surface $\Gamma^{(2)}$ at $\bar{\mathbf{Y}}(\mathbf{X}, t)$.

With these definitions in hand, the convected description of the frictional equations will hereafter be understood to mean prescription of the relationship between $\mathbf{v}_T$ and $\mathscr{T}_T$ (or $\mathscr{T}_T^b$). An important point to be noticed in making this specification is that the base point for both vectors, $\bar{\mathbf{Y}}(\mathbf{X}, t)$, varies with time if attention is fixed on a point $\mathbf{X} \in \Gamma^{(1)}$. This is to be contrasted with the usual situation in nonlinear continuum mechanics, where the base point for a convected object is the same point as the material point in question (consider, for example, large deformation elasticity). This special nature of the contact problem arises because a material point $\mathbf{X} \in \Gamma^{(1)}$ is generally in contact with a different material point of $\Gamma^{(2)}$ at each instant, and must be taken into account when defining appropriate time derivatives of convected quantities.

2.3. *Flow operator and the Lie derivative*

An "appropriate" time derivative in the convected description may be loosely interpreted as one which maintains frame indifference when incorporated into the frictional constitutive theory. As in the theory of plasticity (Simo and Hughes, 1993), a myriad of such "objective rates" exists. It can be shown, however, that all such rates are particular instances of the Lie derivative, an object from differential geometry representing the time derivative of an object as it appears from a particular reference frame. This object will be used to define an objective time derivative here, by appealing to mathematical definitions discussed at some length in Marsden and Hughes (1983), Section 1.6 and Simo *et al.*, (1988).

Referring to Fig. 3, one may begin by defining the mapping $\mathbb{P}: \Gamma^{(1)} \times \mathbf{I} \to \Gamma^{(2)}$, such that $\mathbb{P}(\mathbf{X}, t) = \bar{\mathbf{Y}}(\mathbf{X}, t)$. $\mathbb{P}$ represents a complicated mapping in which both motions $\boldsymbol{\varphi}^{(i)}$ and the projection in (2) are all involved. In practice one would never compute $\mathbb{P}$ directly, but it is useful in defining and envisioning the Lie derivative.

Toward this end, attention is focused on any time $s \in \mathbf{I}$, which is considered to be fixed in this analysis. A point $\bar{\mathbf{Y}} \in \Gamma^{(2)}$ is selected, such that $\bar{\mathbf{Y}} = \mathbb{P}(\mathbf{X}, s)$ for some $\mathbf{X} \in \Gamma^{(1)}$ at the instant in question. In the following development, the particularization of $\mathbb{P}$ to a time $t \in \mathbf{I}$ is written as $\mathbb{P}_t$. For any time $t \in \mathbf{I}$, the following mapping from $\Gamma^{(1)}$ to $\Gamma^{(1)}$ is defined:

$$\boldsymbol{\chi}_{t,s} := \mathbb{P}_t \circ \mathbb{P}_s^{-1} \tag{18}$$

[see again Fig. 3]. Differentiating this mapping with respect to t yields:

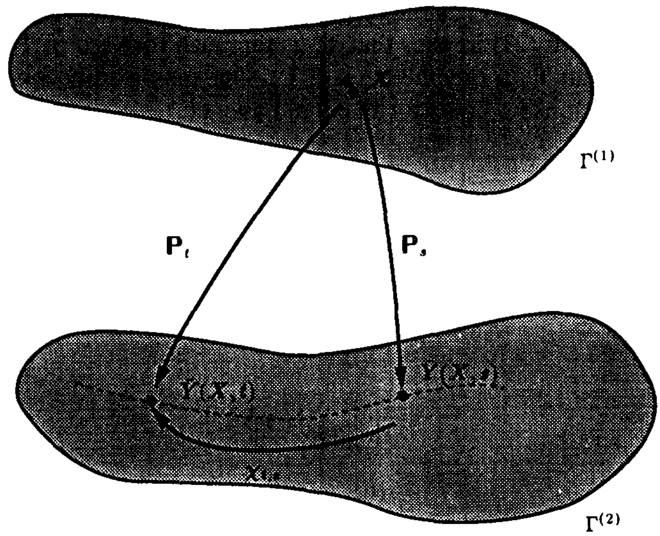

Fig. 3. Definition of the projection mapping $\mathbb{P}$ and the flow operator $\chi_{t,s}$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\chi_{t,s}(\bar{\mathbf{Y}}) &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{P}_t \circ \mathbb{P}_s^{-1})(\bar{\mathbf{Y}}) \\ &= \frac{\partial \mathbb{P}_t}{\partial t}(\mathbb{P}_s^{-1}(\bar{\mathbf{Y}})) \\ &= \boldsymbol{v}_T(\mathbb{P}_t(\mathbb{P}_s^{-1}(\bar{\mathbf{Y}})), t) \\ &= \boldsymbol{v}_T(\chi_{t,s}(\bar{\mathbf{Y}}), t). \end{aligned} \tag{19}$$

In view of (19) and following the classical definition, one can see that for fixed $\bar{\mathbf{Y}} \in \Gamma^{(2)}$ and $s \in \mathbf{I}$, $t \mapsto \chi_{t,s}$ is an integral curve of the convected velocity $\boldsymbol{v}_T$. Further, it is easily seen that:

$$\chi_{s,s}(\bar{\mathbf{Y}}) = \mathbb{P}_s \circ \mathbb{P}_s^{-1}(\bar{\mathbf{Y}}) = \bar{\mathbf{Y}}. \tag{20}$$

Equations (19) and (20) together imply that the collection of all maps $\chi_{t,s}$ defines the flow operator for the convected velocity $\boldsymbol{v}_T$ (Marsden and Hughes, 1983).

The Lie derivative of an object in the convected frame is defined using the flow operator $\chi_{t,s}$. Of most immediate importance is the Lie derivative $\mathscr{L}_v$ of the convected frictional traction $\mathscr{T}^b_T$, which is calculated via

$$\mathscr{L}_v \mathscr{T}^b_T := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} \{\chi^*_{t,s}\mathscr{T}^b_{T_t}\}, \tag{21}$$

where $\chi^*_{t,s}\mathscr{T}^b_{T_t}$ is the pull back induced by $\chi_{t,s}$ and $\mathscr{T}^b_{T_t}$ is the frictional traction at time t. This pull back is readily defined by considering the pull back of the convected $\mathbf{T}_\alpha$ basis (eqn $(8)_1$). As $\chi_{t,s}$ is defined, one has the property that:

$$\mathbf{Y}(\bar{\xi}(\mathbf{X}, t)) = \chi_{t,s}(\mathbf{Y}(\bar{\xi}(\mathbf{X}, s))). \tag{22}$$

Using the chain rule, one finds:

$$\begin{aligned}\mathbf{T}_{\alpha_t} &= \mathbf{Y}_{,\alpha}(\bar{\xi}(\mathbf{X}, t)) \\ &= D\chi_{t,s} \cdot \mathbf{Y}_{,\alpha}(\bar{\xi}(\mathbf{X}, s)) \\ &= D\chi_{t,s} \cdot \mathbf{T}_{\alpha_s} \end{aligned} \tag{23}$$

where $\mathbf{T}_{\alpha_t}$ is the basis at time t, $\mathbf{T}_{\alpha_s}$ is the basis at time s, and where $D\chi_{t,s}$ is the gradient of the mapping $\chi_{t,s}$. Equation (23) implies that $\mathbf{T}_{\alpha_t}$ is the push forward of $\mathbf{T}_{\alpha_s}$ induced by $\chi_{t,s}$ which in turn implies that $\mathbf{T}_{\alpha_s}$ is the pull back of $\mathbf{T}_{\alpha_t}$. One may readily conclude from these facts that the same relationship holds for the dual basis, i.e.:

$$\mathbf{T}_s^{\alpha} = \chi_{t,s}^{*}\mathbf{T}_t^{\alpha}, \tag{24}$$

which is used in conjunction with (21) to obtain

$$\begin{aligned}\mathscr{L}_v\mathscr{T}_T^b &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s}\{\chi_{t,s}^{*}(t_{T_{\alpha_t}}\mathbf{T}_t^{\alpha})\} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s}\{t_{T_{\alpha_t}}\chi_{t,s}^{*}\mathbf{T}_t^{\alpha}\} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s}\{t_{T_{\alpha_t}}\mathbf{T}_s^{\alpha}\} \\ &= \dot{t}_{T_\alpha}\mathbf{T}^{\alpha}.\end{aligned} \tag{25}$$

To conclude, (25) defines a time rate of change of the frictional traction which is frame indifferent due to the use of the Lie derivative (Simo *et al.*, 1988). Furthermore, one notes that the final expression obtained in (25) contains a material time derivative (defined with $\mathbf{X}$ fixed) which acts on the components of $\mathscr{T}_T^b$ only (i.e. no terms containing time derivatives of the base vectors are included). This result holds because $\mathscr{T}_T^b$ is defined in the $\mathbf{T}_\alpha$ basis, which is convected with the point $\bar{\mathbf{Y}}(\mathbf{X}, t)$ as it changes. As will be seen in Section 3, this result makes numerical implementation of the general large deformation theory little more complicated than treatment of the small deformation theory.

2.4. *Physical interpretation*

Before briefly discussing the incorporation of this kinematic framework into a global computational strategy, it is worthwhile to point out the physical significance of the mathematical quantities described in this section. As we have seen, $\boldsymbol{v}_T$ is intimately related to the relative velocity between $\Gamma^{(1)}$ and $\Gamma^{(2)}$ at the contact point $\mathbf{X}$ [see (10) and (11)]. In fact, $(11)_1$ asserts that if body (2) is rigid, $\mathbf{F}_t^{(2)}$ is the identity tensor and $\boldsymbol{v}_T$ is exactly equal to the relative velocity. In the more general event that body (2) deforms during $\mathbf{I}$, one can consider the locus of all points $\bar{\mathbf{Y}}$ for a point $\mathbf{X}$ to define a contact path on $\Gamma^{(2)}$. In the neighborhood of a time $s \in \mathbf{I}$ this path is characterized by the mapping $\chi_{t,s}$. In examining (19) one finds that $\boldsymbol{v}_T$ is the time rate of change of the point $\bar{\mathbf{Y}}$ on this path, since $\chi_{t,s}$ is the integral curve for $\boldsymbol{v}_T$. By examining $\boldsymbol{v}_T$ and its integral over time, then, one obtains measures of relative velocity and sliding distance. These measures are physically motivated, possess required frame indifference, and fit neatly into a computational framework for solving large deformation problems. As a result, numerical implementation of a wide range of frictional constitutive relations, even those obtained in small deformation experiments, is readily made due to the geometric theory discussed.

3. INCORPORATION OF CONSTITUTIVE THEORIES AND NUMERICAL IMPLEMENTATION

In discussing numerical implementation of the frictional theory one should first consider the global equation system to be solved. As shown in Laursen and Simo (1993), the global (Lagrangian) variational principal for the two-body contact system of Fig. 1 takes the following form at any time $t \in \mathbf{I}$:

$$G(\boldsymbol{\varphi}_t, \overset{*}{\boldsymbol{\varphi}}) + G_c(\boldsymbol{\varphi}_t, \overset{*}{\boldsymbol{\varphi}}) = 0. \tag{26}$$

In (26), $\boldsymbol{\varphi}_t$ is the collection of mappings $\boldsymbol{\varphi}_t^{(1)}$ and $\boldsymbol{\varphi}_t^{(2)}, \overset{*}{\boldsymbol{\varphi}}$ are admissible (material) variations, and $G(\boldsymbol{\varphi}_t, \overset{*}{\boldsymbol{\varphi}})$ is the sum of the internal virtual work, the virtual work of the prescribed applied forces and tractions, and (for dynamic problems) the inertial virtual work. $G_c(\boldsymbol{\varphi}_t, \overset{*}{\boldsymbol{\varphi}})$, the contact virtual work, may be shown to be of the following form:

$$G_c(\boldsymbol{\varphi}_t, \overset{*}{\boldsymbol{\varphi}}) = \int_{\Gamma^{(1)}} [t_{N_t} \delta g + t_{T_{\alpha_t}} \delta \xi^{\alpha}] \, \mathrm{d}\Gamma^{(1)}, \tag{27}$$

where δg and $\delta \xi^{\alpha}$ are directional derivatives of g and ξ^{α} in the direction $\overset{*}{\boldsymbol{\varphi}}$. It is important to realize that equations (26) and (27) are obtained exactly, without simplifying geometric assumptions, by beginning with the strong form of the initial/boundary value problem and integrating by parts in the usual manner. The discussion in Laursen and Simo (1993) should be consulted for the details of this derivation.

In any event, equations (26) and (27) represent the global system of equations to be solved for all $t \in \mathbf{I}$. These systems are highly nonlinear in general, and require Newton–Raphson or similar techniques to solve them. Finite element discretization, as well as consistent linearization of the contact equations, is discussed in the aforementioned reference and Laursen (1992), and is therefore omitted here. Of primary interest is the determination of t_{N_t} and $t_{T_{\alpha_t}}$ in (27), which is necessary to solve the global equations. It is for this task that the development of Section 2 is especially pertinent.

As a model constitutive law for the friction perhaps the simplest alternative, Coulomb friction, will be considered. This law can be written in the following form, using the convected description of the previous section:

$$\begin{aligned} \Phi &:= \|\mathscr{T}^b_T\| - \mu t_N \leqslant 0 \\ \boldsymbol{v}^b_T - \zeta \frac{\mathscr{T}^b_T}{\|\mathscr{T}^b_T\|} &= 0 \\ \zeta &\geqslant 0 \\ \Phi\zeta &= 0, \end{aligned} \tag{28}$$

where μ is the coefficient of friction. It has long been recognized (Michalowski and Mroz, 1978) that small deformation Coulomb friction could be written in this general form, but the current formulation allows construction of an analogous expression for large deformations in which frame indifference and geometric meaning are automatically maintained. As written, eqns (28) are unregularized; i.e. no reversible tangential slip is allowed (compare with eqns (3) governing the normal response). Viscoplastic effects and history dependence (of μ, for example) may be incorporated into this framework without difficulty, but are omitted from the present discussion for simplicity.

As was done with the normal response, eqns (28) are usually regularized for computational and/or physical reasons, such that a small amount of elastic tangential relative motion is admitted (Kikuchi and Oden, 1988). This regularization may be introduced by a penalization of the Lie derivative of the frictional traction discussed in Section 2, resulting in the following:

$$\begin{aligned} \Phi &:= \|\mathscr{T}^b_T\| - \mu t_N \leqslant 0 \\ \boldsymbol{v}^b_T - \zeta \frac{\mathscr{T}^b_T}{\|\mathscr{T}^b_T\|} &= \frac{1}{\varepsilon_T} \mathscr{L}_v \mathscr{T}^b_T \\ \zeta &\geqslant 0 \\ \Phi\zeta &= 0. \end{aligned} \tag{29}$$

Notably, the only difference between equations (28) and (29) is in $(29)_2$, where the penalization of the rate replaces a 0. Accordingly, the unregularized problem is only exactly reproduced in the limit where $\varepsilon_T \to \infty$. The structure of these equations is perhaps more readily seen by rearranging them as follows:

$$\begin{aligned} \Phi &:= \|\mathscr{T}_T^b\| - \mu t_N \leqslant 0 \\ \mathscr{L}_v \mathscr{T}_T^b &= \varepsilon_T \left[\mathbf{v}_T^b - \zeta \frac{\mathscr{T}_T^b}{\|\mathscr{T}_T^b\|} \right] \\ \zeta &\geqslant 0 \\ \Phi\zeta &= 0. \end{aligned} \tag{30}$$

The quantity in the brackets of $(30)_2$ represents the rate of elastic slip, which is to be zero in the unregularized problem. This situation is directly analogous to rigid plasticity, with the plastic slip $\zeta(\mathscr{T}_T^b / \|\mathscr{T}_T^b\|)$ being analogous to the plastic strain rate. In the current theory one introduces a tangential stiffness ε_T, which may be either a mathematical penalization or a physical stiffness characterizing the surface in question. In the latter case a direct analogue would be an elastoplastic medium, where ε_T essentially plays the role of the elastic modulus. More sophisticated regularizations, perhaps including elastic anisotropic response, could be obtained by replacing the scalar ε_T in $(30)_2$ by a tensoral regularization (Zmitrowicz, 1992a,b). In any case, eqns (30) show the benefit of the careful definition of $\mathscr{L}_v \mathscr{T}_T^b$ in Section 2; introduction of the regularization including this rate preserves frame indifference of the constitutive equations.

The strongest benefit of using this characterization is made possible by the convected $\mathbf{T}_\alpha$ and $\mathbf{T}^\alpha$ bases. Since no time derivatives of these bases appear in the expression for the Lie derivative, eqns (30) are conveniently written in component form, rendering expressions in which the base vectors do not appear. Performing this operation, and reproducing equation (4) here gives:

$$\left.\begin{aligned} t_N &= \varepsilon_N g^+ \\ \Phi &:= [t_{T_\alpha} M^{\alpha\beta} t_{T_\beta}]^{1/2} - \mu t_N \leqslant 0 \\ \dot{t}_{T_\alpha} &= \varepsilon_T \left[M_{\alpha\beta} \dot{\xi}^\beta - \zeta \frac{t_{T_\alpha}}{[t_{T_\beta} M^{\beta\gamma} t_{T_\gamma}]^{1/2}} \right] \\ \zeta &\geqslant 0 \\ \Phi\zeta &= 0. \end{aligned}\right\} \tag{31}$$

Returning to the original problem of interest, one wishes to solve eqns (26) and (27) for all time $t \in \mathbf{I}$, with the frictional tractions and contact pressure governed by equations (31). This is ordinarily done numerically by dividing $\mathbf{I}$ into a series of subintervals via $\mathbf{I} = \bigcup_{n=0}^{N} [t_n, t_{n+1}]$. The problem is then solved incrementally, by repeatedly stepping forward from a known solution at t_n to find an unknown solution at t_{n+1}. Correspondingly, a discrete time integration algorithm for the frictional stresses can be devised by an appeal to similar integrators used for the theory of plasticity (Giannakopoulos, 1989 or Wriggers *et al.*, 1990). For example, applying a backward Euler integrator to eqns (31) yields:

$$\left.\begin{aligned} t_{N_{n+1}} &= \varepsilon_N g_{n+1}^+ \\ \Phi_{n+1} &:= [t_{T_{n+1_\beta}} M^{\beta\gamma} t_{T_{n+1_\gamma}}]^{1/2} - \mu t_{N_{n+1}} \leqslant 0, \\ t_{T_{n+1_\alpha}} &= t_{T_{n_\alpha}} + \varepsilon_T \left\{ M_{\alpha\beta} [\xi_{n+1}^\beta - \xi_n^\beta] - \zeta \frac{t_{T_{n+1_\alpha}}}{[t_{T_{n+1_\beta}} M^{\beta\gamma} t_{T_{n+1_\gamma}}]^{1/2}} \right\} \\ \zeta &\geqslant 0, \\ \Phi_{n+1}\zeta &= 0, \end{aligned}\right\}, \tag{32}$$

which can be solved using a trial state/return map algorithm. One begins by computing a trial state, which assumes no slip during the increment:

$$\left.\begin{aligned} t_{N_{n+1}} &= \varepsilon_N g_{n+1}^{+} \\ t_{T_{n+1_\alpha}}^{\text{trial}} &= t_{T_{n_\alpha}} + \varepsilon_N M_{\alpha\beta}[\xi_{n+1}^{\beta} - \xi_n^{\beta}] \\ \Phi_{n+1}^{\text{trial}} &= [t_{T_{n+1_\beta}}^{\text{trial}} M^{\beta\gamma} t_{T_{n+1_\gamma}}^{\text{trial}}]^{1/2} - \mu t_{N_{n+1}} \end{aligned}\right\}, \tag{33}$$

after which the slip condition $\Phi_{n+1}^{\text{trial}}$ is checked. The frictional tractions $t_{T_{n+1_\alpha}}$ are then calculated accordingly:

$$t_{T_{n+1_\alpha}} = \begin{cases} t_{T_{n+1_\alpha}}^{\text{trial}} & \text{if } \Phi_{n+1}^{\text{trial}} \leqslant 0 \text{ (stick)}, \\ \mu t_{N_{n+1}} \dfrac{t_{T_{n+1_\alpha}}^{\text{trial}}}{[t_{T_{n+1_\beta}}^{\text{trial}} M^{\beta\gamma} t_{T_{n+1_\gamma}}^{\text{trial}}]^{1/2}} & \text{otherwise (slip).} \end{cases} \tag{34}$$

It is seen that this algorithm is completely displacement-driven, i.e. given the state of deformation at time t_{n+1}, g_{n+1} and ξ_{n+1}^{α} can be determined, which enables calculation of the new frictional tractions by eqns (33) and (34). Furthermore, owing to the convected description employed in the underlying continuum description, the integration scheme p-reserves objectivity without the need for covariant derivatives (as in, for example Glaser, 1992) or elaborate, complicated objective rates (as in Kikuchi and Oden, 1988). In fact, the integration scheme presented in eqns (33)–(34) is little more complicated than that for a small deformation problem, and yet possesses all features necessary for large deformation analysis (i.e. for incremental solution of eqns (26)).

4. NUMERICAL EXAMPLE

In this section a numerical example is briefly presented, to demonstrate the utility of the proposed formulation. The finite element code used in this Lagrangian analysis is FEAP, described at some length in Zienkiewicz and Taylor (1991). The elastoplastic continuum is described using the large deformation continuum formulation in Simo (1988) and Simo (1992), and the frictional description is as described in this paper. The example provides evidence that development of the convected description is not merely an academic exercise, but allows for effective computation when used in conjunction with a carefully formulated numerical strategy for contact (in this case, the formulation described in Laursen and Simo, 1993).

As depicted in Fig. 4(a), the simulation involves the quasistatic deep drawing of a square elastoplastic plate ($K = 10^5$, $G = 10^4$, $\sigma_Y = 100$, H (hardening modulus) $= 1$) through a table-like die with a rounded, cylindrical opening. The drawing is achieved via displacement control of a rigid spherical die, which forces the sheet down into the opening (the lower die is assumed elastic with the same moduli as the sheet, and is fixed on the surface not in contact with the sheet). The coefficient of friction for the workpiece/table and workpiece/loader contacts is taken as $\mu = 0.15$, with the associated penalties being $\varepsilon_N = \varepsilon_T = 10^4$. For symmetry reasons, only one-quarter of the geometry need be modeled in this problem.

Figure 4(b) depicts the final deformed configuration of the sheet and table, with the vertical displacements of the sheet contoured on it. As can be seen, the table deforms very little, with the majority of the deformation taking place in the sheet. Although not indicated in the figure, almost all deformation in the sheet is plastic. Although this example is not rigorous either in its conception (through-the-thickness response is under-resolved, for example) or its presentation, it does provide an example of how the methodology described in this paper can be implemented and used in an actual computation. Future work will explore the numerical and material science issues associated with modeling metal forming problems of this size and larger in an efficient and accurate manner.

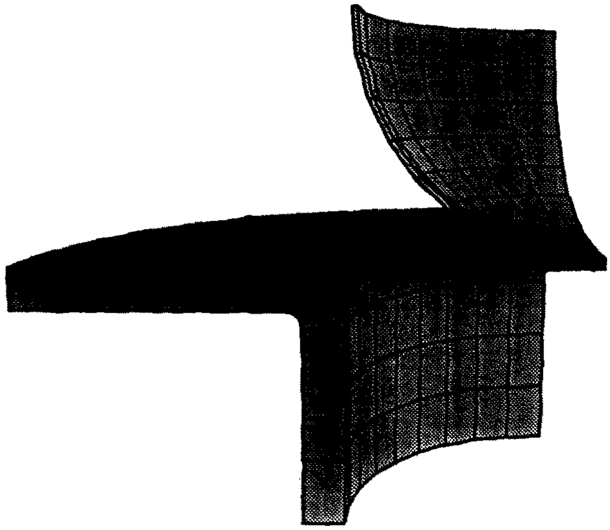

Fig. 4(a). Three dimensional deep drawing metal forming problem, undeformed configuration.

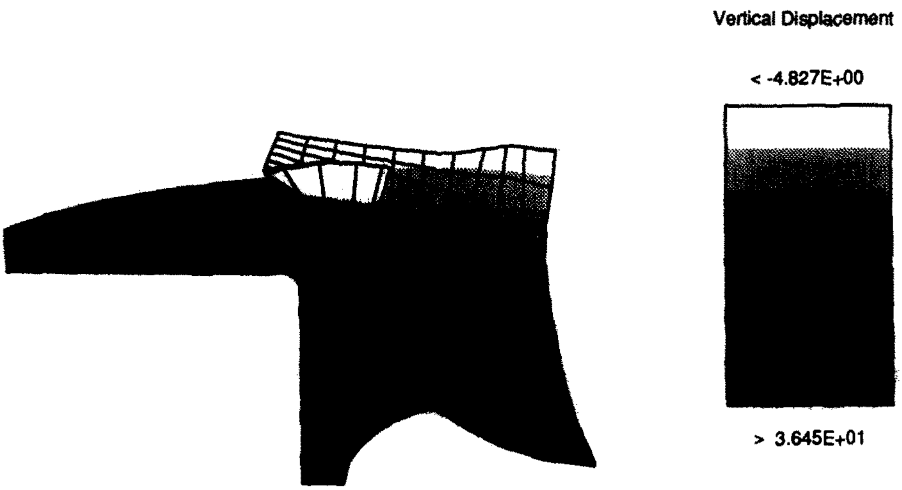

Fig. 4(b). Final deformed configuration for the sheet forming problem.

5. CONCLUSIONS

In this paper, a convected description of large deformation frictional processes has been developed from a continuum mechanical point of view. It has been shown that this description is readily interpreted both mathematically and physically, making adaptation of frictional laws to the large deformation setting a relatively simple exercise. The resulting formulation is amenable to convenient numerical implementation and is in fact quite general, even though only demonstrated in this paper for simple Coulomb friction. It is expected that future work will investigate the incorporation of more general theories of friction into the kinematic framework provided in this presentation.

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